SHENTON COLLEGE

Mathematics Specialist: Units 3 & 4

Test 1: Complex Numbers

Working Time: 50 minutes
Total marks: 60 marks

Formula sheet provided
No notes permitted
No ClassPad (nor any other calculator) permitted

$\frac{\quad}{60}$

Name: MARKING KEY   Teacher: ALFONSI   MOORE

Note: Please read all questions carefully, and note that when a part of a question is worth more than two marks, adequate and clear working out is required for full marks.

1. Given that $z_1 = \sqrt{2}\text{cis}\left(\frac{\pi}{3}\right)$, $z_2 = \sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$, $z_3 = -2i$ and $z_4 = -1 - \sqrt{3}i$, determine

[1 + 2 + 2 + 3 + 3 = 11 marks]

(a) z_4 in polar form.

$z_4 = 2\,\text{cis}\left(\frac{-2\pi}{3}\right)$ ✓

(b) $z_2 + z_3$ in Cartesian form.

$z_2 + z_3 = (1-i) + (-2i) = 1-3i$

✓ z_2 in polar form
✓ z_2+z_3

(c) the product $z_1 z_2$ in polar form.

$z_1 z_2 = \sqrt{2}\sqrt{2}\,\text{cis}\left(\frac{\pi}{3} + \left(-\frac{\pi}{4}\right)\right) = 2\,\text{cis}\left(\frac{\pi}{12}\right)$

✓ modulus
✓ argument

(d) the quotient $\frac{z_1}{z_3}$ in polar form.

$\frac{z_1}{z_3} = \frac{\sqrt{2}\,\text{cis}\left(\frac{\pi}{3}\right)}{2\,\text{cis}\left(-\frac{\pi}{2}\right)} = \frac{\sqrt{2}}{2}\,\text{cis}\left(\frac{\pi}{3} - \left(-\frac{\pi}{2}\right)\right) = \frac{\sqrt{2}}{2}\,\text{cis}\left(\frac{5\pi}{6}\right)$

✓ z_3 in polar form
✓ modulus
✓ argument

(e) $(z_4)^6$ in Cartesian form.

$(z_4)^6 = \left(2\,\text{cis}\left(-\frac{2\pi}{3}\right)\right)^6 = 2^6\,\text{cis}\left(6\cdot\left(-\frac{2\pi}{3}\right)\right) = 64\,\text{cis}(-4\pi)$
$= 64$

✓ modulus via deMoivre's
✓ argument via deMoivre's
✓ answer in Cartesian form

2. If $z = r\text{cis}\theta$, express the following in cis form in terms of r and/or θ:

[1 + 1 + 1 + 1 = 4 marks]

(a) $\bar{z}$

$= r\,\text{cis}(-\theta)$ ✓

(b) $z\bar{z}$

$= r^2\,\text{cis}(0)$ ✓

(allow $= r^2$)

(c) iz^2

$= r^2\,\text{cis}\left(2\theta + \frac{\pi}{2}\right)$ ✓

(d) $\dfrac{1-i}{1+i}z$

$= r\,\text{cis}\left(\theta - \frac{\pi}{2}\right)$ ✓

3. Arithmetic operations on complex numbers can be described geometrically in terms of *translations, rotations, reflections* and *enlargements* in the complex plane.

Explain the sequence of transformations which correspond to taking a complex number z and transforming it to $2i(\bar{z} - i)$.

[4 marks]

$z \to \bar{z}$: a reflection in the real axis, followed by ✓

$\bar{z} \to \bar{z} - i$: a translation by one unit down, followed by ✓

$\bar{z} - i \to 2i(\bar{z} - i)$: an anticlockwise rotation by $\frac{\pi}{2}$ about the origin and ✓

an enlargement by a factor of 2 about the origin. ✓

note: -1 per error or omission regarding order.

4. Consider the equation $z^4 = 2\sqrt{3} + 2i$.

[5 + 1 = 6 marks]

(a) Determine all of the solutions to this equation, giving your answers in polar form and in terms of their prinicpal argument.

$z^4 = 4\,\text{cis}\left(\frac{\pi}{6}\right)$ ✓ RHS to polar form

$z = 4^{\frac{1}{4}}\,\text{cis}\left(\frac{\pi/6 + 2\pi K}{4}\right)$ $K = -2, -1, 0, 1$ ✓ sets up (or clearly states rotation) $\Delta\theta = \pi/2$.

$= \sqrt{2}\,\text{cis}\left(\frac{\pi + 12\pi K}{24}\right)$

hence $z_0 = \sqrt{2}\,\text{cis}\left(\frac{\pi}{24}\right)$ ✓ first sol$^{\underline{n}}$

$z_1 = \sqrt{2}\,\text{cis}\left(\frac{13\pi}{24}\right)$

$z_2 = \sqrt{2}\,\text{cis}\left(-\frac{23\pi}{24}\right)$ ✓ next three sol$^{\underline{n}}$s

$z_3 = \sqrt{2}\,\text{cis}\left(-\frac{11\pi}{24}\right)$

✓ uses principal argument

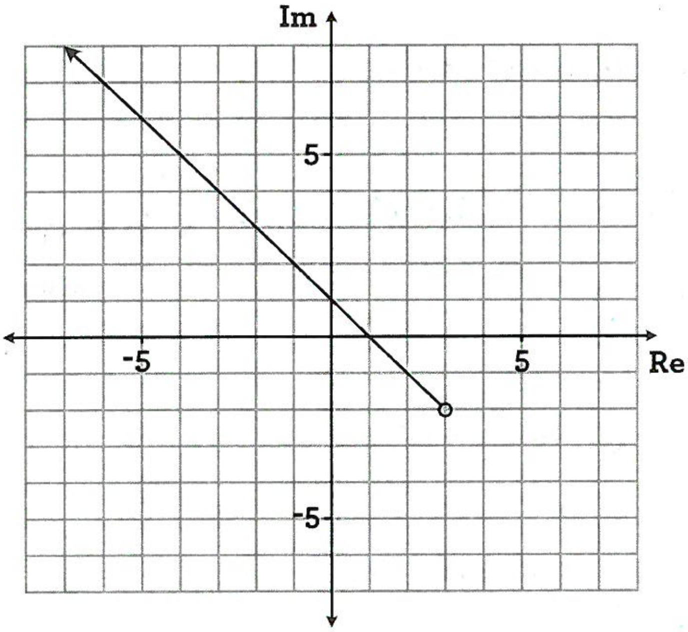
(b) Explain in a single sentence why it is unsurprising that none of the solutions of the polynomial $z^4 - (2\sqrt{3} + 2i) = 0$ are complex conjugate pairs.

This is a polynomial with a complex coefficient, so the Complex Conjugate Root Theorem does not apply. ✓ or equivalent statement

5. Express, using set notation, the locus of z in each of the following diagrams.

[3 + 3 = 6 marks]

(a)



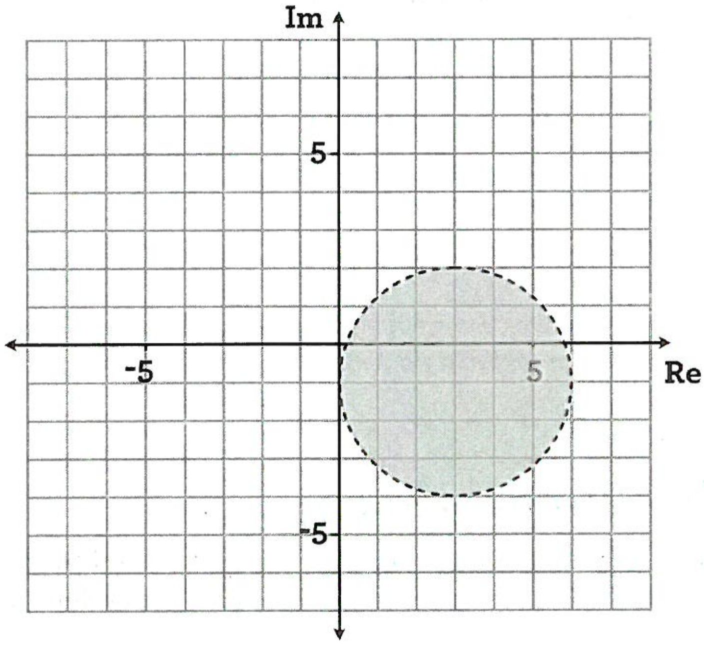
$\{z : \arg(z - (3 - 2i)) = \frac{3\pi}{4}\}$

✓ correct type

✓ offset

✓ angle

(b)



$\{z : |z - (3 - i)| < 3\}$

✓ correct type and uses <

✓ centre

✓ radius

6. Use the Argand diagrams provided to sketch the regions in the complex plane defined by the following loci.

[3 + 3 + 3 = 9 marks]

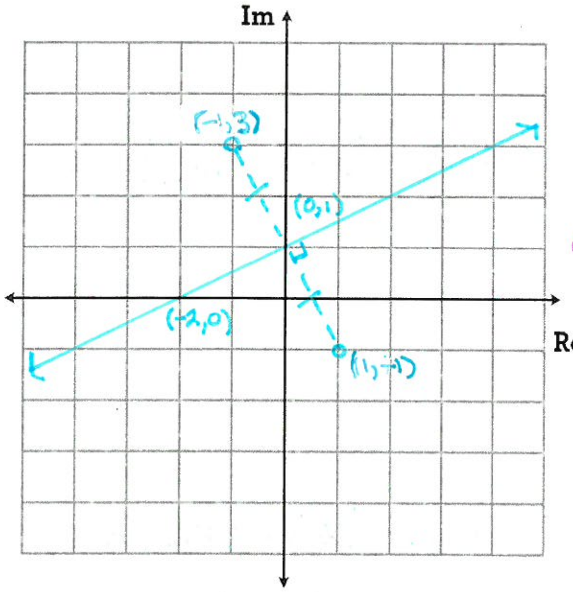
(a) $\{z : |z - 1 + i| = |z + 1 - 3i|\}$

$\{z : |z - (1-i)| = |z - (-1+3i)|\}$

✓ recognises type (e.g., by marking line segment)

✓ line ($\perp$ bisector)

✓ sufficient points marked to be unambiguous (i.e., endpoints, $\perp$, ~~tot~~ or, axis intercepts)



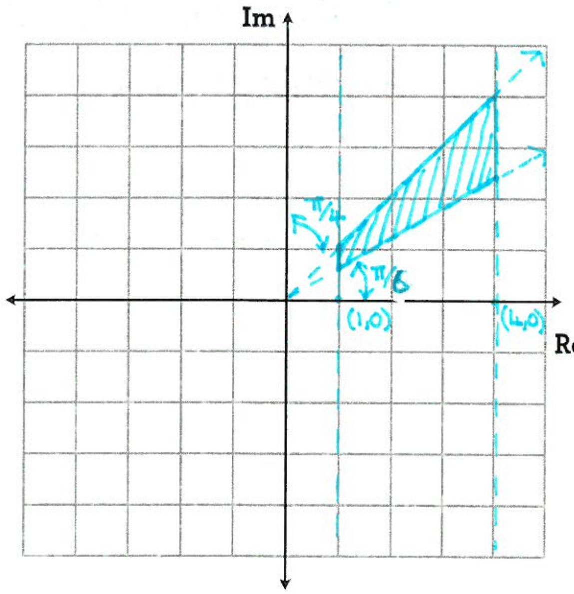
note: require arrowheads

(b) $\{z : \frac{\pi}{6} \le \arg(z) \le \frac{\pi}{4}\} \cap \{z : 1 \le \text{Re}(z) \le 4\}$

✓ angular range

✓ horizontal range

✓ boundary, shading, no additional inclusions



(c) $\{z : \text{Im}(z) = |z - i|\}$

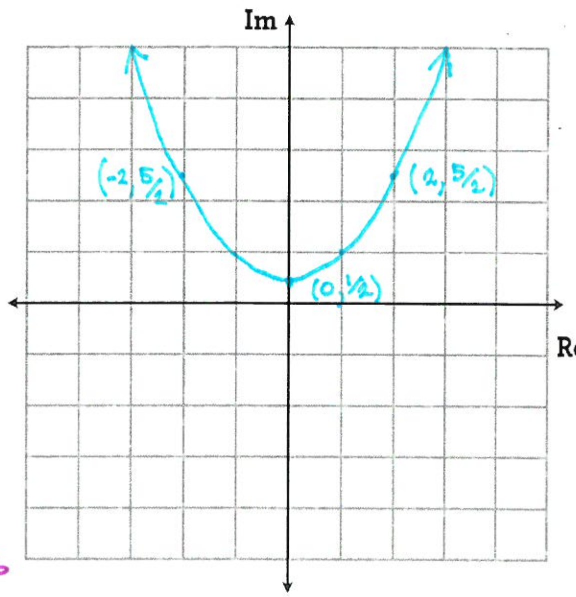
Let $z = x+iy$

then $y = |x + i(y-1)|$

$y^2 = x^2 + (y-1)^2$

$y^2 = x^2 + y^2 - 2y + 1$

$y = \dfrac{x^2+1}{2}$ ✓ establishes this relation



✓ recognises and sketches a parabola.

✓ sufficient points marked to be unambiguous.

7. Consider $P(z) = z^3 + az^2 + bz + c$ with $a, b, c \in \mathbb{R}$. Two of the roots of $P(z) = 0$ are -2 and $(-3 + 2i)$.

[2 + 3 = 5 marks]

(a) Write $P(z)$ in fully-factored form. (I.e., express $P(z)$ as the product of its linear factors.)

$P(z) = (z - (-3+2i))(z - (-3-2i))(z+2)$

✓ recognises conjugate root

✓ writes in fully-factored form

(b) Hence, determine the values of the coefficients a, b and c.

$$P(z) = (z^2+6z+13)(z+2)$$
$$= z^3+6z^2+13z+2z^2+12z+26$$
$$= z^3+8z^2+25z+26$$

✓ expands complex factors

✓ fully expands and collects

hence, $a=8$, $b=25$ and $c=26$.

✓ notes a, b, c

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

8. Given that

$$\frac{\sin 4\theta}{\sin \theta} = A\cos^3\theta + B\cos\theta \qquad (\sin\theta \neq 0)$$

[5 + 1 = 6 marks]

(a) use de Moivre's theorem with $n = 4$ to determine the values of A and B.

$\text{cis}4\theta = (\text{cis}\theta)^4$

$\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$

✓ sets up deMoivre's and carries out binomial expansion

$= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4$

$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$

$\Rightarrow \sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ ✓ equates Im

so, $\dfrac{\sin 4\theta}{\sin\theta} = 4\cos^3\theta - 4\cos\theta\sin^2\theta$ ✓ divides through

$= 4\cos^3\theta - 4\cos\theta(1-\cos^2\theta)$

$= 4\cos^3\theta - 4\cos\theta + 4\cos^3\theta$

✓ applies Pythagorean identity, expands and collects

$= 8\cos^3\theta - 4\cos\theta$

Hence, $A = 8$ and $B = -4$ ✓ notes A and B

(b) Hence, determine the limiting value of $\dfrac{\sin 4\theta}{\sin\theta}$ as θ approaches zero (i.e., $\lim\limits_{\theta\to 0}\dfrac{\sin 4\theta}{\sin\theta}$).

$\lim\limits_{\theta\to 0}\dfrac{\sin 4\theta}{\sin\theta} = \lim\limits_{\theta\to 0}(8\cos^3\theta - 4\cos\theta)$

$= 8 - 4 = 4$ ✓

9. Consider $u = \text{cis}\left(\frac{\pi}{4}\right)$, one of the 8th roots of unity, and $v = \text{cis}\left(\frac{\pi}{3}\right)$, one of the 6th roots of unity.

[2 + 2 + 2 + 3 = 9 marks]

(a) Mark and label the positions of u, u^2, u^4 and u^6 as well as v, v^2, v^4 and v^6 on the Argand diagram at the right.

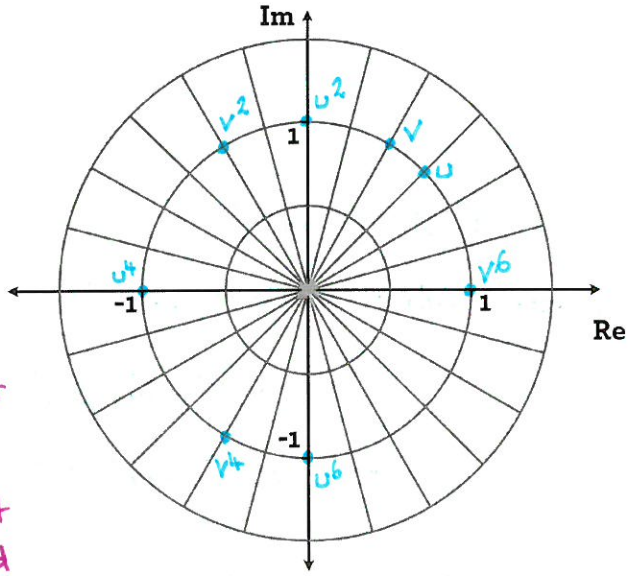
$u^n = \text{cis}\left(\frac{n\pi}{4}\right)$

$v^n = \text{cis}\left(\frac{n\pi}{3}\right)$

so...

✓ all u^n correct and labelled

✓ all v^n correct and labelled



(b) For what values of $m \in \mathbb{Z}$ is u^m purely real and negative?

$\frac{m\pi}{4} = (2k+1)\pi \quad , k \in \mathbb{Z}$ ✓ notes this

$m = 8k+4 \quad , k \in \mathbb{Z}$ ✓ simplifies

(c) For what values of $n \in \mathbb{Z}$ is v^n purely real and positive?

$\frac{n\pi}{3} = 2k\pi$ ✓ notes this

$n = 6k \quad , k \in \mathbb{Z}$ ✓ simplifies

(d) What is the smallest value of $p \in \mathbb{Z}^+$ such that the product $u^{2-p}v^{p-7}$ is purely real?

$(2-p)\frac{\pi}{4} + (p-7)\frac{\pi}{3} = k\pi \quad , k \in \mathbb{Z}$ ✓ LHS ($\times$, + args) ✓ RHS ($k\pi$ for purely real)

$3(2-p) + 4(p-7) = 12k$

$p = 12k + 22$

hence the smallest $p \in \mathbb{Z}^+$ is 10 (when $k=-1$). ✓ determines smallest $p \in \mathbb{Z}^+$.

[END OF TEST]